Atiyah-Patodi-Singer index from the domain-wall Dirac operator

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This talk is based on

[FOY]

H. Fukaya, T. Onogi, SY, Phys. Rev. D 96, 125004 (2017)

[FFMOYY]

H. Fukaya, M. Furuta, S. Matsuo, T. Onogi, M. Yamashita, SY, arXiv:1910.01987

String theorist

Lattice gauge theorists

Mathematicians

Introduction

Index theorem of Dirac operators (Atiyah-Singer)

A theorem on the number of solutions

$$D\psi = 0 \qquad D := \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

Index theorem

$$\overbrace{n_{+} - n_{-}}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

Index theorem appears in various situations in physics

- Number of generations in compactification.

- Anomaly. $\psi' = e^{i\alpha\gamma_5}\psi$

$$\int D\psi' D\bar{\psi}' = \int D\psi D\bar{\psi} e^{i\alpha \mathrm{Ind}(D)}$$

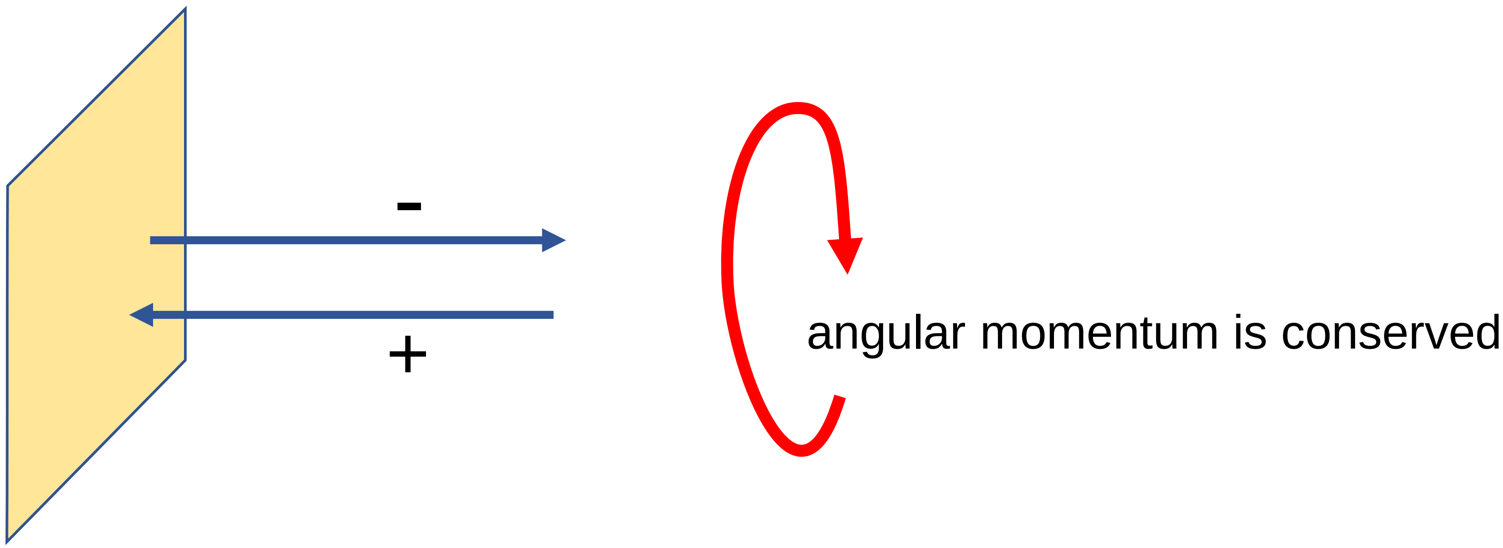
- Topological material (Discussed later).

Any index theorem with **boundary?**

- Number of generations in compactification with **boundary?**
- Anomaly with **boundary?**
- Topological material with **boundary?**

Difficulty of index with boundary

If we impose **local** and **Lorents (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer (APS) boundary condition

[Atiyah, Patodi, Singer 75]

Abandon the locality and preserve the chirality.

Eg. 4 dim $x^4 \geq 0$

$A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose

$$(A + |A|)\psi|_{x^4=0} = 0$$

APS index theorem [Atiyah, Patodi, Singer 75]

$$\mathrm{Ind}(D) = \frac{\eta(iD^{3D})}{2} + \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

“eta invariant” (discussed later)

Any application to physics?

cf. [Alvarez-Gaume, Della Pietra, Moore 85]

Topological insulator.

- 4 dim massive fermion.
- CP symmetry is imposed.

Two distinct “phases” according to the sign of mass.

$$S = \int d^4x \bar{\psi}(D \pm M)\psi$$

One is **trivial** and the other is “**topological insulator**”

Topological insulator phase with boundary has **massless edge modes**.

Topological insulator and APS index

[Witten 15]

$$S = \int d^4x \bar{\psi}(D \pm M)\psi$$

Partition function

$$Z = \int D\psi D\bar{\psi} e^{-S}$$

APS index

Trivial phase	Topological insulator
$Z = \lvert Z \rvert$	$Z = \lvert Z \rvert (-1)^{\mathrm{Ind}(D)}$

However APS setup and topological insulator look quite different!

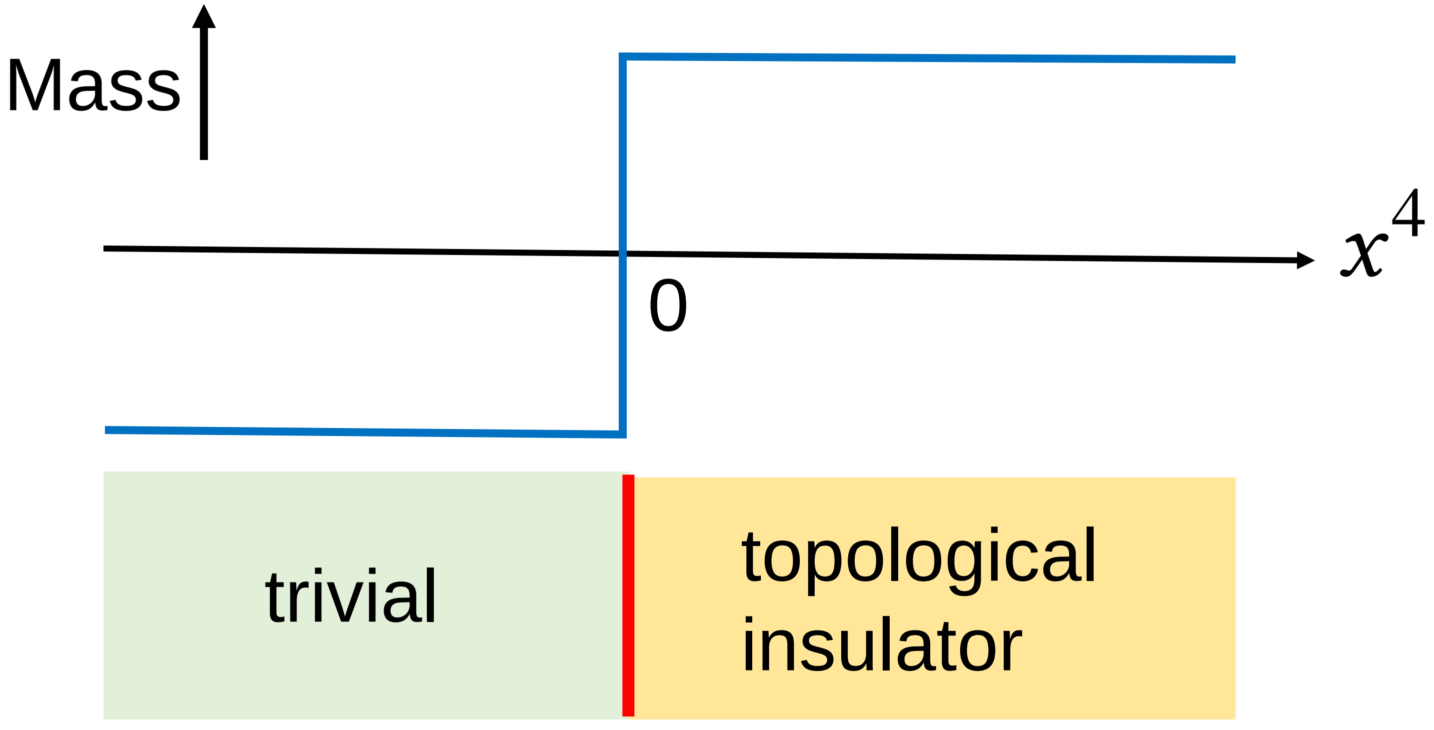
See also [Witten, Yonekura 19]

APS		Top. Ins.
Massless fermion	⟷	Massive fermion
Non-local B.C.	⟷	Local B.C.
No massless edge localized mode	⟷	Massless edge localized mode exists

We want to clarify this relation starting from a setup more close to topological insulator.

Domain-wall setup

(interface, defect, …)



In this setup, we define “domain-wall index” $\mathcal{I}$ and show

- Partition function is written as

$$Z = |Z|(-1)^{\mathcal{I}}$$

- Calculate $\mathcal{I}$ using Fujikawa’s method and obtain

$$\mathcal{I} = \frac{\eta(iD^{3D})}{2}\Big|_{x^4=0} + \frac{1}{32\pi^2}\int_{x^4>0} d^4x\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

(= APS index)

- Give a mathematically rigorous proof of (domain-wall index) = (APS index)

Massive fermion and index without boundary

Massive fermion in 4 dim coupled to background gauge field

$$D := \gamma^\mu(\partial_\mu + iA_\mu)$$

$$S = \int d^4x \bar{\psi}(D + M)\psi \qquad M > 0$$

Consider the phase of the partition function

$$Z = \int D\psi D\bar{\psi} e^{-S} = \det(D + M)$$

But this is divergent …

We employ Pauli-Villars regularization

$$Z = \frac{\det(D + M)}{\det(D - \Lambda)} \qquad \Lambda > 0$$

CP symmetry ⇒ Z is real.

$$Z = |Z|(-1)^J$$

Let us find J

$$Z = \frac{\det(D+M)}{\det(D-\Lambda)}$$

$$= \frac{\det i\gamma_5(D+M)}{\det i\gamma_5(D-\Lambda)} = \frac{\det iH}{\det iH_{PV}}$$

$$H := \gamma_5(D+M) \qquad H_{PV} := \gamma_5(D-\Lambda)$$

Both of them are **Hermitian** operators.

$$\det iH = \prod_{\lambda:\text{eigenvalues of} H} i\lambda$$

$$H := \gamma_5(D+M) \qquad H_{PV} := \gamma_5(D-\Lambda) \qquad Z = \frac{\det iH}{\det iH_{PV}}$$

$$\det iH = \prod_{\lambda:\text{eigenvalues of} H} i\lambda$$

$$i\lambda = |\lambda| \exp(i\frac{\pi}{2}\text{sign}\lambda)$$

$$\det iH = |\det iH| \exp(i\frac{\pi}{2}\sum_{\lambda}\text{sign}\lambda)$$

$$\eta(H) := \sum_{\lambda}\text{sign}\lambda$$

regularized by zeta function regularization

“Eta invariant”

$$Z = |Z|(-1)^J$$

$$J = \frac{\eta(H)}{2} - \frac{\eta(H_{PV})}{2}$$

We can calculate $\eta(H)$ by eg. Fujikawa's method.

$$\eta(H) = \frac{1}{32\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

$$= \mathrm{Ind}(D)$$

※ This is independent of M as far as M>0

※ We can also obtain

$$\eta(H_{PV}) = -\frac{1}{32\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

For topological insulator

$$Z = |Z|(-1)^J$$

$$J = \frac{\eta(H)}{2} - \frac{\eta(H_{PV})}{2} = \eta(H) = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

$$= \mathrm{Ind}(D)$$

For trivial phase $M \longrightarrow -M$

$$Z = \frac{\det(D-M)}{\det(D-\Lambda)} = |Z|(-1)^J$$

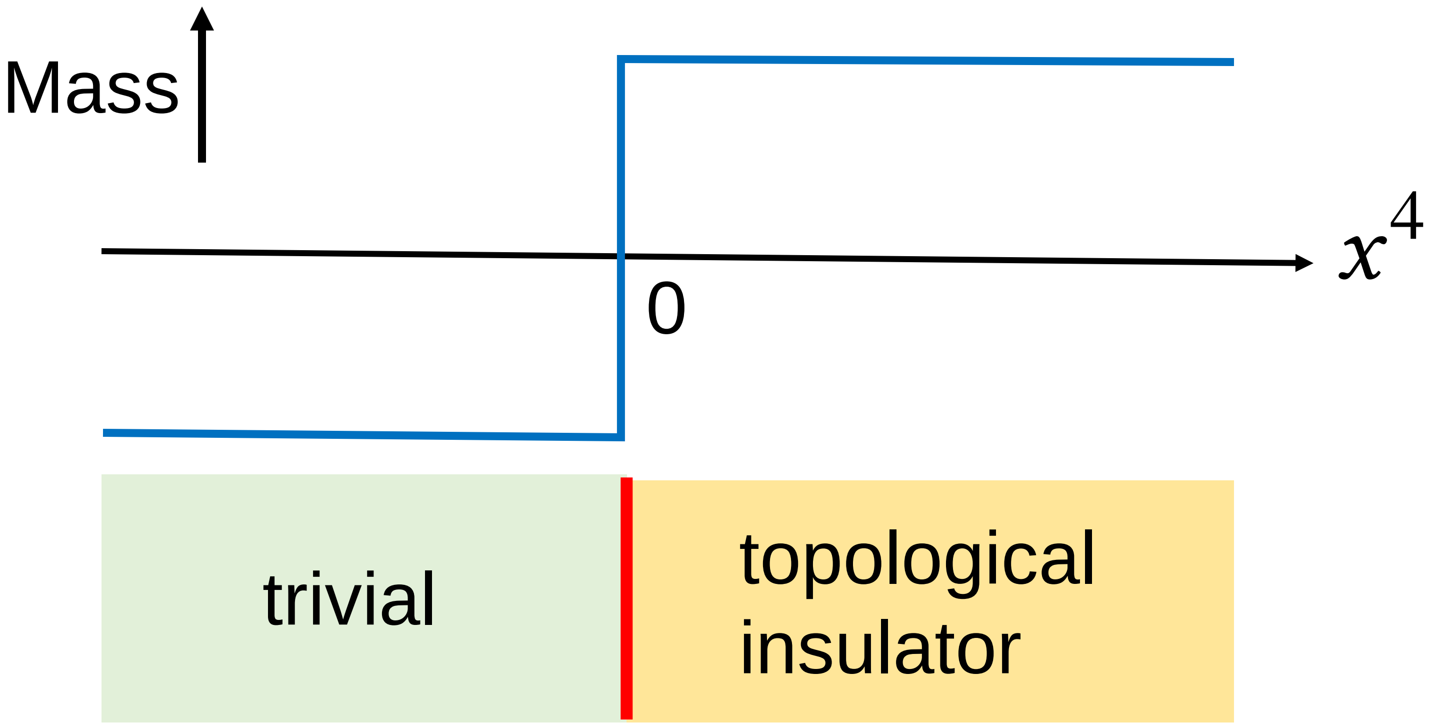
$$J = \frac{\eta(\gamma_5(D-M))}{2} - \frac{\eta(H_{PV})}{2} = 0$$

Summary

$$Z = |Z|(-1)^J$$

Index appear in the phase of the partition function of a massive fermion.

Domain-wall Dirac operator



$$S = \int d^4x \bar{\psi}(D + M\epsilon(x^4))\psi$$

$$\epsilon(x^4) = \begin{cases} -1, & (x^4 < 0) \\ +1, & (x^4 > 0) \end{cases}$$

By the same argument

$$Z = \frac{\det(D + M\epsilon(x^4))}{\det(D - \Lambda)} = |Z|(-1)^{\mathcal{I}}$$

$$\mathcal{I} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$$

$$H_{DW} := \gamma_5(D + M\epsilon(x^4))$$

Let us call this integer “domain-wall index”

We calculated $\eta(H_{DW})$ by Fujikawa's method.

Result:

$$\eta(H_{DW}) = \underline{\eta(iD^{3D})|_{x^4=0}} + \frac{1}{32\pi^2}\int d^4x\,\underline{\epsilon(x^4)}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

Domain-wall index

$$\mathcal{I} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$$

$$= \frac{\eta(iD^{3D})}{2}|_{x^4=0} + \frac{1}{32\pi^2}\int_{x^4>0} d^4x\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

=(APS index in $x^4 > 0$)

Mathematical Proof

Our observation

[Fukaya, Onogi, SY]

(APS index in $x^4 > 0$)= $\frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$

is a mathematically rigorous conjecture!

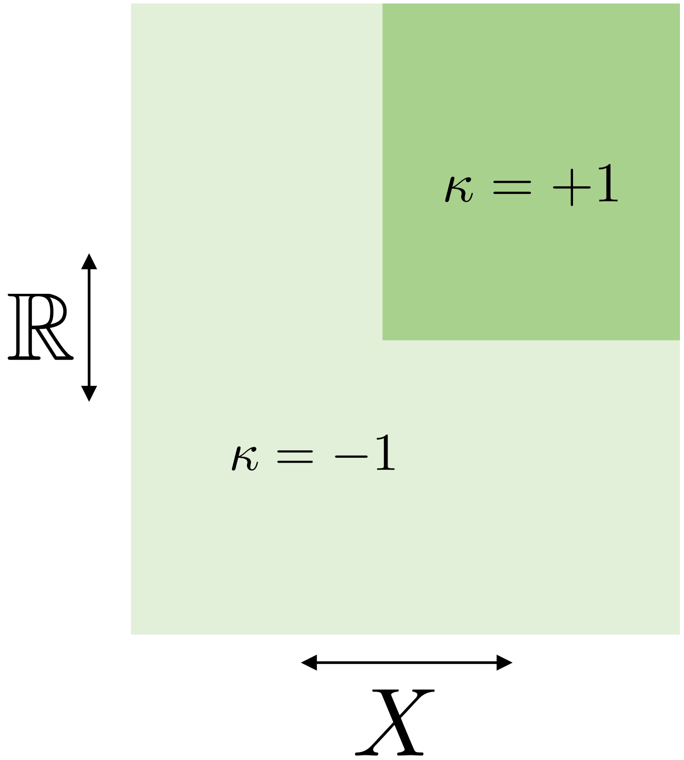
(noticed by Mikio Furuta)

We gave a mathematically rigorous proof of a generalized version
(general even dimensions, curved background,) . [Fukaya, Furuta, Matsuo, Onogi, Yamashita, SY]

Sketch of the proof

Go to 5 dimensions $\mathbb{R} \times X$

4-dim, closed, curved, gauge fields



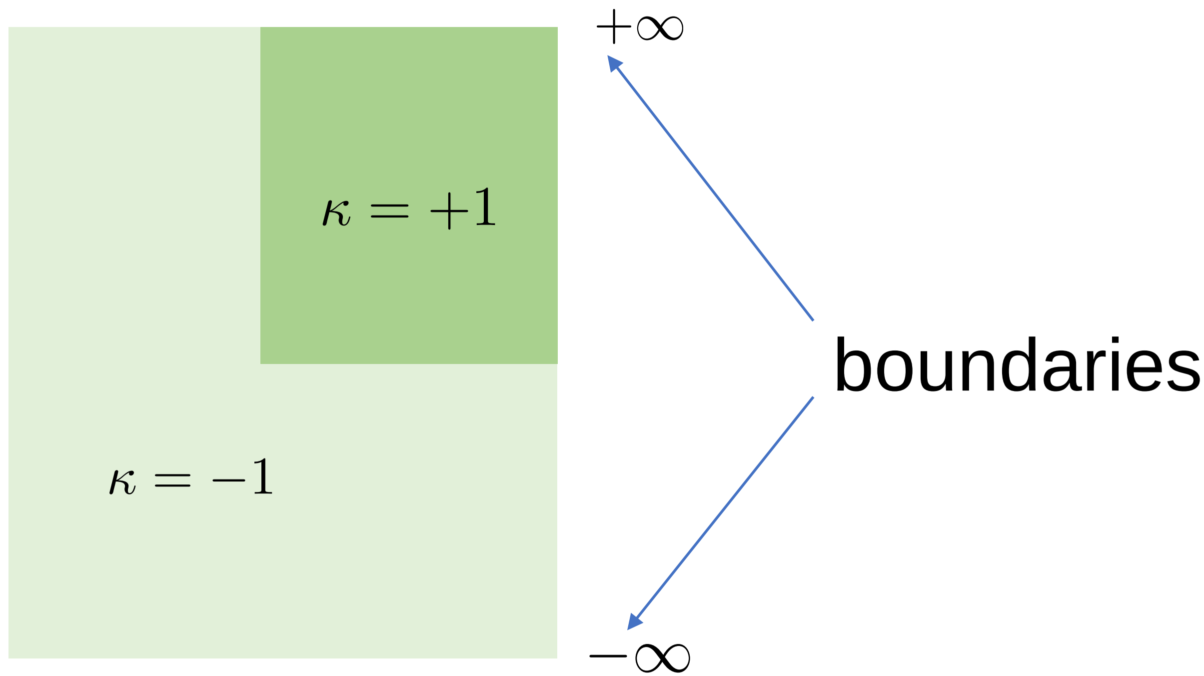
$$D_{5D} := \begin{pmatrix} 0 & \gamma_5(D + M\kappa) + \partial_s \\ \gamma_5(D + M\kappa) - \partial_s & 0 \end{pmatrix}$$

Let us evaluate $\mathrm{Ind}(D_{5D})$ by two different ways

Sketch of the proof

$$D_{5D} := \begin{pmatrix} 0 & \gamma_5(D+M\kappa)+\partial_s \\ \gamma_5(D+M\kappa)-\partial_s & 0 \end{pmatrix}$$

1. Use APS index theorem



H_{DW} H_{PV}

$$\mathrm{Ind}(D_{5D}) = \frac{\eta(\gamma_5(D+M\kappa)|_{+\infty})}{2} - \frac{\eta(\gamma_5(D+M\kappa)|_{-\infty})}{2}$$

No bulk term!

since 5-dim

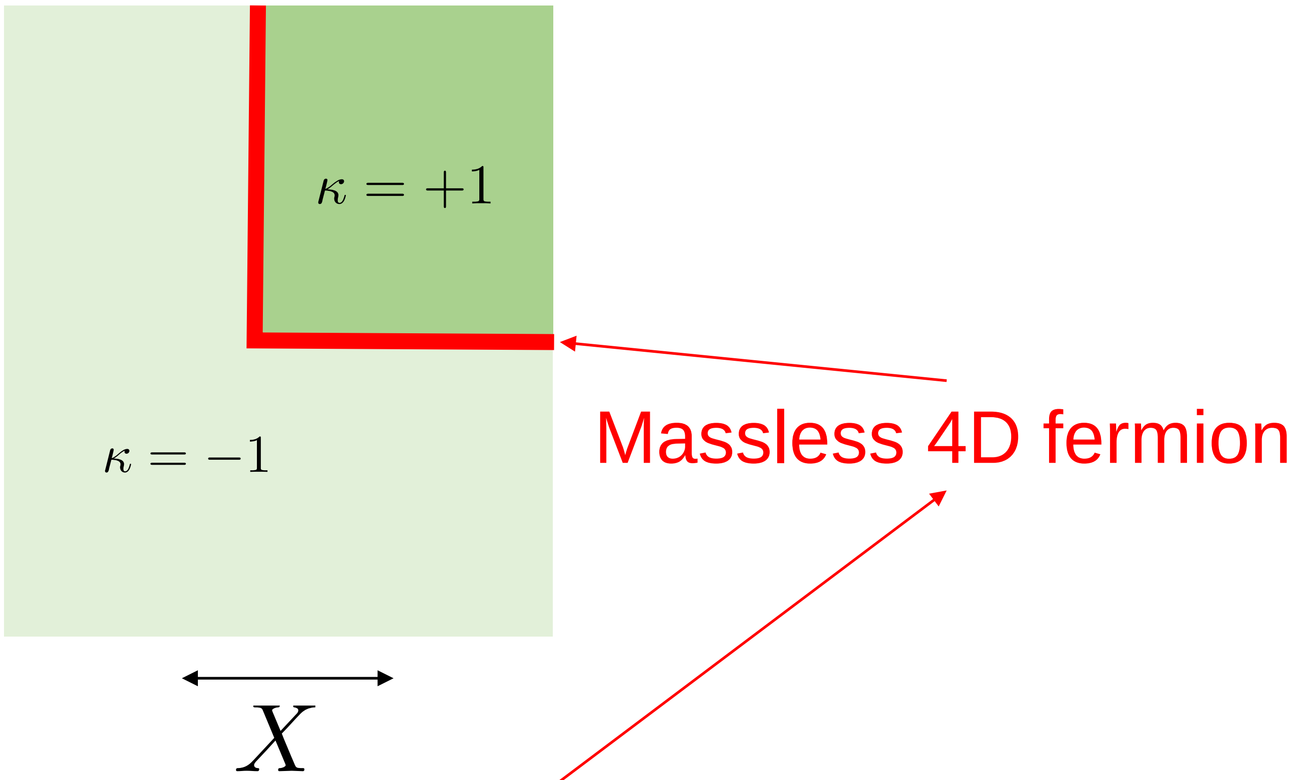
$$= \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$$

Sketch of the proof

$$D_{5D} := \begin{pmatrix} 0 & \gamma_5(D+M\kappa)+\partial_s \\ \gamma_5(D+M\kappa)-\partial_s & 0 \end{pmatrix}$$

2. Use localization

Adding infinite throat is equivalent to imposing APS boundary condition



$$\mathrm{Ind}(D_{5D}) =$$ (Index of this fermion)

$$= \mathrm{Ind}(D)$$ (APS index)

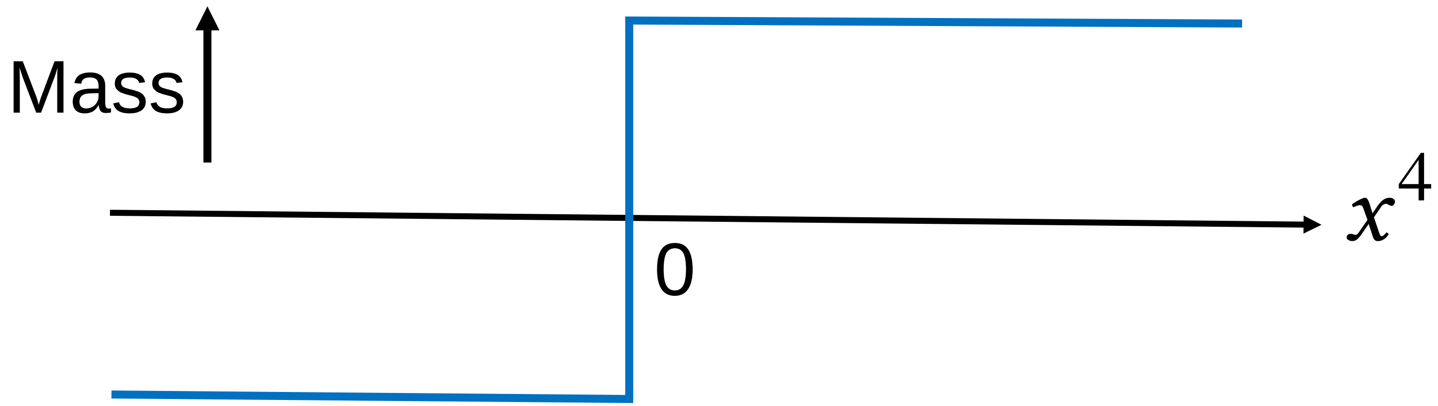
$$\mathrm{Ind}(D_{5D}) \overset{1.}{=} \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$$

$$\overset{2.}{=} \mathrm{Ind}(D)$$ (APS index)

This equality is what we wanted to proove

Summary

Domain-wall setup (close to topological insulator)



- We define domain-wall index $\mathcal{I} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$
- It appears in the phase of the partition function
- We have prooved that

  (domain-wall index) = (APS index)